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Inner ideals, compact tripotents and Čebyšëv subtriples of JB*-triples and C*-algebras

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Abstract The aim of this note is to study Čebyšëv JB*-subtriples of general JB*-triples. It is established that if F is a non-zero Čebyšëv JB*-subtriple of a JB*-triple E, then exactly one of the following statements holds:

(a) F is a rank one JBW*-triple with $\dim(F) \geq 2$ (i.e. a complex Hilbert space regarded as a type 1 Cartan factor). Moreover, F may be a closed subspace of arbitrary dimension and E may have arbitrary rank;
(b) $F = \mathbb{C}e$, where e is a complete tripotent in E;
(c) E and F are rank two JBW*-triples, but F may have arbitrary dimension;
(d) F has rank greater or equal than three and $E = F$.

Keywords Čebyšëv/Chebyshev subspace · JB*-triples · Čebyšëv/Chebyshev subtriple · von Neumann algebra · C*-algebra · Brown-Pedersen quasi-invertibility · spin factor

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1 Introduction

It is known that certain problems on operator algebras are more feasible when the algebra under study is a von Neumann algebra (i.e. a C*-algebra which is also a dual Banach space). For example, A.G. Robertson gave in [35] a complete description of one-dimensional Čebyšëv subspaces, and of finite dimensional Čebyšëv hermitian subalgebras with dimension bigger than 1 of a general von Neumann algebra. Concretely, for a non-zero element x in a von Neumann algebra M, the subspace $\mathbb{C}x$ is a Čebyšëv subspace of M if and only if there is a projection p in the center of M such that px is left invertible in pM and $(1-p)x$ is right invertible in $(1-p)M$ (cf. [35, Theorem 1]). A finite dimensional *-subalgebra N of an infinite dimensional von Neumann algebra M with dim $(N) > 1$ never is a Čebyšëv subspace of M (see [35, Theorem 6]).

Two years later A.G. Robertson and D. Yost proved in [36, Corollary 1.4] that an infinite dimensional C*-algebra A admits a finite dimensional *-subalgebra B which is also a Čebyšëv subspace in A if and only if A is unital and $B = \mathbb{C}1$. The results proved by Robertson and Yost were complemented by G.K. Pedersen, who shows that if A is a C*-algebra without unit and B is a Čebyšëv C*-subalgebra of A, then $A = B$ (compare [34, Theorem 4]).

We recall that a subspace V of a Banach space X is called a *Čebyšëv* (*Chebyshev*) *subspace* of X if for each $x \in X$ there exists a unique point $\pi_V(x) \in V$ such that dist$(x, V) = \left\|x - \pi_V(x)\right\|$. Throughout this note the symbol $\pi_V(x)$ will denote the best approximation of an element x in X in a Čebyšëv subspace V of X. For more information on Čebyšëv and best approximation theory we refer to the monograph [37].

Similar benefits to those obtained working with von Neumann algebras re-appear when studying Čebyšëv subspaces of Ternary Rings of Operators (TRO's) of a given von Neumann algebra, or when exploring Čebyšëv JBW*-subtriples of a given JBW*-triple (see Sect. 2 for definitions). In a previous paper, we establish the following description of Čebyšëv JBW*-subtriples of a JBW*-triple.

Theorem 1 [26, Theorem 13] Let N be a non-zero Čebyšëv JBW*-subtriple of a JBW*-triple M. Then exactly one of the following statements holds:

(a) N is a rank one JBW*-triple with dim $(N) \geq 2$ (i.e. a complex Hilbert space regarded as a type 1 Cartan factor). Moreover, N may be a closed subspace of arbitrary dimension and M may have arbitrary rank;
(b) $N = \mathbb{C}e$, where e is a complete tripotent in M;
(c) N and M have rank two, but N may have arbitrary dimension;
(d) N has rank greater or equal than three and $N = M$. □

We refer to [21, Preliminaries] and [11, Example 2.5.31] for the definition of Cartan factors.

The question whether in the above theorem JBW*-triples and subtriples can be replaced with JB*-triples and subtriples remained as an open problem. The techniques employed in [26] rely heavily on the rich geometric properties of JBW*-triples. In this note we study this problem in the more general setting of JB*-triples. We combine here new arguments involving inner ideals and compact tripotents in the bidual of a JB*-triple. The main result of this note shows that the conclusion of the above Theorem 1 also holds when N is a JB*-subtriple of a general JB*-triple M (see Theorem 11).

Among the new results proved in this note we also establish that a Čebyšëv C*-subalgebra B (respectively, a Čebyšëv JB*-subtriple) of a C*-algebra A with rank$(B) \geq 3$ coincides with the whole A (see Corollary 10).

2 Preliminaries

The multiple attempts to understand a Riemann mapping theorem type for complex Banach spaces of dimension bigger or equal than 2, led some mathematicians to the study of bounded symmetric domains (compare [10,23,24,33] and [29]). The definite answer was given by W. Kaup, who showed the existence of a set of algebraic-geometric-analytic axioms which determine a class of complex Banach spaces, the class of JB*-triples, whose open unit balls are bounded symmetric domains, and every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB*-triple; in this way, the category of all bounded symmetric domains with base point is equivalent to the category of JB*-triples.

A *JB*-triple* is a complex a Banach space E with a continuous triple product $(a, b, c) \mapsto \{a, b, c\}$, which is bilinear and symmetric in the external variables and conjugate linear in the middle one and satisfies:

(a) (Jordan identity)

$$L(x, y)\{a, b, c\} = \{L(x, y)a, b, c\} - \{a, L(y, x)b, c\} + \{a, b, L(x, y)c\},$$

for all $x, y, a, b, c \in E$, where $L(x, y) : E \to E$ is the linear mapping given by $L(x, y)z = \{x, y, z\}$;

(b) For each $x \in E$, the operator $L(x, x)$ is hermitian with non-negative spectrum;

(c) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in E$.

Given an element a in a JB*-triple E, the symbol $Q(a)$ will denote the conjugate linear map on E defined by $Q(a)(x) := \{a, x, a\}$.

The class of JB*-triples includes all C*-algebras when the latters are equipped with the triple product given by

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a). \tag{2.1}$$

The space $B(H, K)$ of all bounded linear operators between complex Hilbert spaces, although rarely is a C*-algebra, is a JB*-triple with the product defined in (2.1). In particular, every complex Hilbert space is a JB*-triple. Thus, the class of JB*-triples is strictly wider than the class of C*-algebras.

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual [1]). The triple product of every JBW*-triple is separately weak* continuous (cf. [1]). The second dual, E^{**}, of a JB*-triple, E, is a JBW*-triple with a certain triple product extending the product of E (cf. [12]).

Let E be a JB*-triple. An element $e \in E$ is called a *tripotent* if $\{e, e, e\} = e$. For each tripotent $e \in E$, the eigenspaces of the operator $L(e, e)$ induce a decomposition (called *Peirce decomposition*) of E in the form

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0, 1, 2$, $E_i(e) = \{x \in E : L(e, e)(x) = \frac{i}{2}x\}$ (compare [33, Theorem 25]). The natural projections of E onto $E_i(e)$ will be denoted by $P_i(e)$. It is known that this decomposition satisfies the following multiplication rules:

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e),$$

if $i - j + k \in \{0, 1, 2\}$ and is zero otherwise. In addition,

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.$$

A tripotent e in E is called *complete* (respectively, *minimal*) if the equality $E_0(e) = 0$ (respectively, $E_2(e) = \mathbb{C}e \neq \{0\}$) holds.

The connections between JB*-triples and JB*-algebras are very deep. Every JB*-algebra is a JB*-triple under the triple product defined by

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*. \tag{2.2}$$

The Peirce 2-subspace $E_2(e)$ is a JB*-algebra with product $x \circ_e y := \{x, e, y\}$ and involution $x^{\sharp_e} := \{e, x, e\}$.

Let a be an element in a JB*-triple E. It is known that the JB*-subtriple, E_a, generated by a, identifies with some $C_0(L_a)$ where $\|a\| \in L_a \subseteq [0, \|a\|]$ with $L_a \cup \{0\}$ compact, and the element a is associated with a positive generating element in $C_0(L_a)$ (cf. [29, 1.15]). The above identification lifts to that of the bidual of the JB*-triple generated by a with a commutative von Neumann algebra the identity element of which, $r(a)$, is called the range tripotent of a in E^{**}, and we note that a is positive in $E_2^{**}(r(a))$ (see [13, §3]).

A (closed) subtriple I of a JB*-triple E is said to be a *triple ideal* or simply an *ideal* of E if $\{E, E, I\} + \{E, I, E\} \subseteq I$. If we only have $\{I, E, I\} \subseteq I$ we say that I is an *inner ideal* of E. Following standard notation, given an element a in E, we denote by $E(a)$ the norm-closure of $Q(a)(E) = \{a, E, a\}$ in E. It is known that $E(a)$ is precisely the norm-closed inner ideal of E generated by a (cf. [6]). The identity $2Q(a, b) = Q(a + b) - Q(a) - Q(b)$ $(a, b \in E)$ implies that a JB*-subtriple I of E is an inner ideal if and only if I contains $E(a)$ for all a in I. It is also known that

$$E(a)^{**} = \overline{E(a)}^{\sigma(E^{**}, E^*)} = E_2^{**}(r(a)) \tag{2.3}$$

(see [6, Proposition 2.1 and comments prior to it]).

If e and a are contained in a JB*-subtriple F of E we have

(*) $F_k(e) = E_k(e)$ if and only if $E_k(e)$ is contained in F;
(**) $F(a) = E(a)$ if and only if $E(a)$ is contained in F.

The first of these equivalences is immediate upon application of the projection $P_k(e)$. As for $(**)$, $F(a)$ is contained in $E(a)$, by definition, and if F contains $E(a)$ then F^{**} contains $E(a)^{**} = E_2^{**}(r(a))$ so that $F(a)^{**} = E(a)^{**}$, by (a), implying the result. A similar argument shows that $F(a)$ is the intersection of F with $E(a)$.

We recall that two elements a, b in a JB*-triple E are *orthogonal* (written as $a \perp b$) if $L(a, b) = 0$ (see [7, Lemma 1] for several equivalent reformulations). Given a subset $M \subseteq E$, we write $M_E^{\perp}$ (or simply $M^{\perp}$) for the (orthogonal) annihilator of M defined by $M_E^{\perp} = \{y \in E : y \perp x, \forall x \in M\}$. If $e \in E$ is a tripotent, then $\{e\}_E^{\perp} = E_0(e)$, and $\{a\}_E^{\perp} = (E^{**})_0(r(a)) \cap E$, for every $a \in E$ (cf. [8, Lemma 3.2]).

It is known that

$$\|a + b\| = \max\{\|a\|, \|b\|\}, \tag{2.4}$$

whenever a and b are orthogonal elements in a JB*-triple (cf. [20, Lemma 1.3(a)]). A subset $S \subseteq E$ is said to be *orthogonal* if $0 \notin S$ and $x \perp y$ for every $x \neq y$ in S. The minimal cardinal number r satisfying $card(S) \leq r$ for every orthogonal subset $S \subseteq E$ is called the *rank* of E (and will be denoted by $r(E)$). Given a tripotent $e \in E$, the rank of the Peirce 2-subspace $E_2(e)$ will be called the rank of e.

We shall also make use of a natural partial order defined on the set of tripotents (see Corollary 1.7 and comments preceding it in [20]). Given two tripotents e, u in a JB*-triple E, we say that $e \leq u$ if $u - e$ is a tripotent in E with $u - e \perp e$.

We finally, recall that an element x in a JB*-triple E is called *Brown-Pedersen quasi-invertible* (BP quasi-invertible for short) if there exists $y \in E$ such that $B(x, y) = 0$ (cf. [27]), where $B(x, y)$ denotes the Bergmann operator $B(x, y) = I_E - 2L(x, y) + Q(x)Q(y)$. Theorems 6 and 11 in [27] prove that an element x in E is Brown-Pedersen quasi-invertible if, and only if, x is von Neumann regular in the sense of [9,15,30] and its range tripotent is an extreme point of the closed unit ball of E, equivalently, there exists a complete tripotent $v \in E$ such that x is positive and invertible in $E_2(v)$. In particular, every extreme point of the closed unit ball of E is BP quasi-invertible. The symbol E_q^{-1} will denote the set of BP quasi-invertible elements in E.

3 Čebyšëv subtriples of JB*-triples

The following auxiliary results were established in [26, §3]

Proposition 2 [26, Propositions 9 and 10] Let F be a Čebyšëv JB*-subtriple of a JB*-triple E. Suppose e is a non-zero tripotent in F. Then the following statements hold:

(a) $E_0(e) = F_0(e)$, and consequently, every complete tripotent in F is complete in E.
(b) If $\{e\}_F^{\perp} \neq 0$, then $E_2(e) = F_2(e)$. □

We continue, in this paper, our study on Čebyšëv subtriples of general JB*-triples. The first part of the next proposition owes much to the arguments developed by Pedersen in [34, Lemma 4].

Proposition 3 *Let a belong to a Čebyšëv JB*-subtriple F of a JB*-triple E. The following statements hold:*

(a) If a is not BP quasi-invertible we have $F(a) = E(a)$;
(b) If F contains no BP quasi-invertible elements we have that F is an inner ideal of E.

Proof (a) Since a is in $F \backslash F_q^{-1}$ we have two possibilities either a is not von Neumann regular or a is von Neumann regular and its range tripotent is not an extreme point of the closed unit ball of F. We deal with each case separately. We can assume that $\|a\| = 1$. Suppose first that a is not von Neumann regular. Then 0 is a non-isolated point in the triple spectrum L_a of a. We know that in this case, $0, 1 \in L_a \subseteq [0, 1]$, with L_a compact. Regarding F_a as a commutative C*-algebra by its identification with $C_0(L_a)$ in the standard way, given a positive ε, as in [34, Lemma 4] we can choose positive norm one elements x, y and z in F_a such that $\{y, x, y\} = x$ such that y (and hence, x) is orthogonal to z and $\|a - x\| < \varepsilon$. In addition, we have that $r(x)$ is orthogonal to $y - r(x)$ (in F^{**}). In particular, $Q(y)^2$ must restrict to the identity map on $E(x)$. We claim that

$$F \text{ contains } E(x).$$

To see which, let w belong to $E(x)$. With $c = \pi_F(w)$ we have c lies in F and $\|w - Q(y)^2(c)\| = \|Q(y)^2(w-c)\| \leq \|w - c\|$, so that $c = Q(y)^2(c)$ by uniqueness of best approximation, which further implies that c is orthogonal to z. If c is not equal to w choose any positive real $\lambda \leq \|w - c\|$ to give

$$\|w - (\lambda z + c)\| = \|(w - c) - \lambda z\| = \max\{\|w - c\|, \lambda\} = \|w - c\|,$$

contradicting unique approximation. Thus $w = c$ and so belongs to F, which proves the claim. Since the above argument is true for all positive ε we deduce that $E(a)$ is contained in F, as desired.

If a is von Neumann regular, so that its annihilator is nonzero and $r(a)$ lies in F then, since a belongs to $E_2(r(a))$, F must contain $E(a)$ by Proposition 2(b). The statement (a) follows from (∗).

(b) Is immediate from (a) and the comments in page 4. □

Lemma 4 *Let F be a JB*-subtriple of a JB*-triple E. Suppose F contains no BP quasi-invertible elements (equivalently, $\partial_e(F_1) = \emptyset$). Then for each $e \in \partial_e(E_1)$ we have* $\mathrm{dist}(e, F) = 1$. *If F is a Čebyšëv subspace of E, we have $\pi_F(e) = 0$, for every e as above.*

Proof Suppose we can find $x \in F$ satisfying $\|e - x\| < 1$. Then

$$\|e - P_2(e)(x)\| = \|P_2(e)(e - x)\| \leq \|e - x\| < 1.$$

Since e is the unit element of the JB*-algebra $E_2(e)$, we deduce that $P_2(e)(x)$ is an invertible element in $E_2(e)$. Lemma 2.2 in [25] implies that x is BP quasi-invertible in E, and hence BP quasi-invertible in F, which is impossible. The second statement is clear because $\mathrm{dist}(e, F) = 1 = \|e\|$.

We establish now an strengthened version of Proposition 2. The result is inspired by an argument in [34, Theorem 4].

Proposition 5 *Let F be a Čebyšëv JB*-subtriple of a JB*-triple E. Suppose that a is a non-zero element in F. Then $\{a\}_E^{\perp} \subseteq F$.*

Proof Arguing by contradiction, we suppose the existence of an element $x \in \{a\}_E^{\perp} \setminus F$. Fix a real number t and consider the automorphism of F (and E) given by $S_t = \exp(itL(a, a))$. It is clear that $S_t(\pi_F(x)) = \pi_F(S_t(x))$, for every $t \in \mathbb{R}$. Having in mind that $a \perp x$ it follows that $L(a, a)^n(x) = 0$, for every natural n, which shows that $S_t(x) = x$ for every real t. Therefore

$$\pi_F(x) = \pi_F(x) + itL(a, a)(\pi_F(x)) + \sum_{n=2}^{\infty} \frac{i^n t^n}{n!} L(a, a)(\pi_F(x)).$$

Differentiating at $t = 0$ we conclude that $L(a, a)(\pi_F(x)) = 0$, or equivalently $a \perp \pi_F(x)$ (cf. [7, Lemma 1]).

We have proved that $a \perp x, \pi_F(x)$. Therefore $\pi_F(x) + \mu a \in F$, for every $\mu \in \mathbb{C}$ and, by orthogonality,

$$0 < \mathrm{dist}(x, F) = \|x - \pi_F(x)\| = \max\{\|x - \pi_F(x)\|, \|\mu a\|\} = \|x - \pi_F(x) - \mu a\|,$$

for every $\mu \in \mathbb{C}$ with $\|\mu a\| \leq \|x - \pi_F(x)\|$, contradicting the uniqueness of the best approximation of x in F. □

We recall that a tripotent u in the bidual of a JB*-triple E is said to be *open* when $E_2^{**}(u) \cap E$ is weak* dense in $E_2^{**}(u)$ (see [14]). A tripotent e in E^{**} is said to be *compact-G_δ* (relative to E) if there exists a norm one element a in E such that e coincides with $s(a)$, the support tripotent of a (see [14]). A tripotent e in E^{**} is said to be *compact* (relative to E) if there exists a decreasing net (e_λ) of tripotents in E^{**} which are compact-G_δ with infimum e, or if e is zero.

Closed and bounded tripotents in E^{**} were introduced and studied in [16] and [17]. A tripotent e in E^{**} such that $E_0^{**}(e) \cap E$ is weak* dense in $E_0^{**}(e)$ is called *closed* relative to E. When there exists a norm one element a in E such that $a = e + P_0(e)(a)$, the tripotent e is called *bounded* (relative to E) (cf. [16]). Theorem 2.6 in [16] (see also [19, Theorem 3.2]) asserts that a tripotent e in E^{**} is compact if, and only if, e is closed and bounded.

Corollary 6 *Let F be a Čebyšëv JB*-subtriple of a JB*-triple E. Let e be a tripotent in F^{**} satisfying that $F_2^{**}(e) \cap F \neq \{0\}$. Then $\{e\}_E^{\perp} = \{x \in E : x \perp e\} = E \cap E_0^{**}(e) \subseteq F$. Furthermore, if e is closed in E^{**} relative to E we also have $E_0^{**}(e) = F_0^{**}(e)$.*

Proof By hypothesis, the set $F \cap F_2^{**}(e)$ is non-zero, thus, there exists a non-zero element $a \in F \cap F_2^{**}(e)$. It is easy to check that $\{e\}_E^{\perp} \subseteq \{a\}_E^{\perp}$, and the latter is contained in F by Proposition 5.

We have already proved that $\{e\}_E^{\perp} = E \cap E_0^{**}(e) \subseteq F$, which implies that $E \cap E_0^{**}(e) = F \cap F_0^{**}(e)$. Since e is closed in E^{**}, we can assure that

$$E_0^{**}(e) = \overline{E \cap E_0^{**}(e)}^{\sigma(E^{**},E^*)} = \overline{F \cap F_0^{**}(e)}^{\sigma(E^{**},E^*)} \subseteq F_0^{**}(e) \subseteq E_0^{**}(e).$$

□

Corollary 7 *Let F be a Čebyšëv JB*-subtriple of a JB*-triple E. Let a be a non-zero element in F and let $r(a)$ denote the range tripotent of a in F^{**}. Suppose that $\{a\}_F^{\perp} \neq \{0\}$. Then $E_0^{**}(r(a)) = F_0^{**}(r(a))$.*

Proof Proposition 3(a) implies that $E(a) = F(a)$. Therefore $E(a)^{**} = F(a)^{**}$ is an open JB*-subtriple of E^{**} relative to E in the sense employed in [16,18,19]. Proposition 3.3 in [19] (or [16, Corollary 2.9]) implies that every compact tripotent in $F(a)^{**}$ is compact in E^{**}. Let us take a compact tripotent $e \in F(a)^{**}$ satisfying that $e \leq r(a)$ and $F_2^{**}(e) \cap F \neq \{0\}$. Since e is compact, and hence closed in E^{**} (cf. [16, Theorem 2.6]), Corollary 6 proves that $E_0^{**}(e) = F_0^{**}(e)$. Finally, it is easy to see that, since $r(a) \geq e$, $E_0^{**}(r(a)) \subseteq E_0^{**}(e) = F_0^{**}(e) \subseteq F^{**}$, and hence $E_0^{**}(r(a)) = F_0^{**}(r(a))$.

We turn now our focus to the Peirce 1-subspace associated with a range tripotent. For this purpose we state the following technical lemma.

Lemma 8 *Let e and f be orthogonal tripotents in a JB*-subtriple F of a JB*-triple E such that $E_0(e)$, $E_0(f)$ and $E_2(e + f)$ are contained in F. Then $E = F$.*

Proof It is sufficient to show that the Peirce 1-subspace of $e + f$ is contained in F (since the other two Peirce subspaces of $e + f$ are automatically in F). It is known (elementary calculation) that $E_1(e + f)$ is always contained in $E_0(e) + E_0(f)$, which is contained in F by hypothesis, giving the result. □

We can establish now our first main result on Čebyšëv JB*-subtriples of a general JB*-triple.

Theorem 9 *Let F be a Čebyšëv JB*-subtriple of a JB*-triple E. Suppose F has rank greater or equal than three. Then $E = F$.*

Proof Since F has rank greater or equal than three, we can find mutually orthogonal norm-one elements a, b, c in F.

Proposition 3(a) yields $E(a + b) = F(a + b)$. From (2.3) we conclude that

$$E_2^{**}(r(a + b)) = \overline{E(a + b)}^{\sigma(E^{**},E^*)} = \overline{F(a + b)}^{\sigma(F^{**},F^*)} \subseteq F^{**}.$$

Corollary 7 now implies that $E_0^{**}(r(a)) = F_0^{**}(r(a))$ and $E_0^{**}(r(b)) = F_0^{**}(r(b))$. We deduce from Lemma 8 that $E^{**} = F^{**}$. Finally, as a consequence of the Hahn-Banach Theorem, it is easy to check that $E = F$, as desired. □

Corollary 10 *Let B be a Čebyšëv JB*-subtriple of a C*-algebra A. Suppose B has rank greater or equal than three. Then $A = B$.*

It remains to study Čebyšëv JB*-subtriples of rank smaller or equal than two. In this case, the conclusion will follow from the main result in [26] and the studies about finite rank JB*-triples developed in [5] and [2].

Theorem 11 *Let F be a non-zero Čebyšëv JB*-subtriple of a JB*-triple E. Then exactly one of the following statements holds:*

(a) F is a rank one JBW*-triple with dim $(F) \geq 2$ (i.e. a complex Hilbert space regarded as a type 1 Cartan factor). Moreover, F may be a closed subspace of arbitrary dimension and E may have arbitrary rank;
(b) $F = \mathbb{C}e$, where e is a complete tripotent in E;
(c) E and F are rank two JBW*-triples, but F may have arbitrary dimension;
(d) F has rank greater or equal than three and $E = F$.

Proof If F has rank ≥ 3, Theorem 9 implies that $E = F$. We may therefore assume that F has rank ≤ 2. It follows from [5, Proposition 4.5 and comments at the beggining of §4] (see also [2, §3]) that F is reflexive. So, F is a reflexive JBW*-triple of rank ≤ 2.

We shall adapt next the arguments in the proof of [26, Theorem 13], providing a simplified argument. Every JBW*-triple admits an abundant collection of complete tripotents or extreme points of its closed unit ball (cf. [3, Lemma 4.1] and [32, Proposition 3.5] or [11, Theorem 3.2.3]). Thus, we can find a complete tripotent e in F. There are only two possibilities: either e is minimal in F or e has rank two in F.

When e is rank two in F, we can write $e = e_1 + e_2$ with e_1, e_2 mutually orthogonal minimal tripotents in F. Proposition 2 proves that $E_2(e_j) = F_2(e_j) = \mathbb{C}e_j$, $E_0(e_j) = F_0(e_j)$, and $E_0(e_1 + e_2) = F_0(e_1 + e_2) = \{0\}$, which proves that e_1 and e_2 are minimal tripotents in E, e is complete in E, and E is a rank-2 JBW*-triple.

We finally assume that e is minimal and complete in F. If dim $(F) = 1$, then $F = \mathbb{C}e$, and we are in case (b), otherwise we are in case (a). □

It should be remarked here that Remark 7 in [26] provides an example of an infinite dimensional rank-one Čebyšëv JB*-subtriple of a JB*-triple, while [26, Remark13] gives an example of a rank-one Čebyšëv JB*-subtriple of a rank-n JBW*-triple, where n is an arbitrary natural number.

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References

1. Barton, T., Timoney, R.M.: Weak*-continuity of Jordan triple products and its applications. Math. Scand. **59**, 177–191 (1986)
2. Becerra Guerrero, J., López Pérez, G., Peralta, A.M., Rodríguez-Palacios, A.: Relatively weakly open sets in closed balls of Banach spaces, and real JB*-triples of finite rank. Math. Ann. **330**(1), 45–58 (2004)
3. Braun, R.B., Kaup, W., Upmeier, H.: A holomorphic characterization of Jordan-C*-algebras. Math. Z. **161**, 277–290 (1978)
4. Bunce, L.J.: Norm preserving extensions in JBW*-triple preduals. Quart. J. Math. Oxf. **52**(2), 133–136 (2001)
5. Bunce, L.J., Chu, C.-H.: Compact operations, multipliers and Radon-Nikodym property in JB*-triples. Pac. J. Math. **153**, 249–265 (1992)
6. Bunce, L.J., Chu, C.H., Zalar, B.: Structure spaces and decomposition in JB*-triples. Math. Scand. **86**, 17–35 (2000)
7. Burgos, M., Fernández-Polo, F.J., Garcés, J.J., Martínez Moreno, J., Peralta, A.M.: Orthogonality preservers in C*-algebras, JB*-algebras and JB*-triples. J. Math. Anal. Appl. **348**, 220–233 (2008)
8. Burgos, M., Garcés, J.J., Peralta, A.M.: Automatic continuity of biorthogonality preservers between weakly compact JB*-triples and atomic JBW*-triples. Studia Math. **204**(2), 97–121 (2011)
9. Burgos, M., Kaidi, A., Morales, A., Campoy, A.M., Peralta, M.I., Ramírez, M.: Von Neumann regularity and quadratic conorms in JB*-triples and C*-algebras. Acta Math. Sin. **24**(2), 185–200 (2008)
10. Cartan, E.: Sur les domaines bornés homogenes de l'esplace de n variables complexes. Abh. Math. Sem. Univ. Hambg. **11**, 116–162 (1935)
11. Chu, C.H.H.: ordan Structures in Geometry and Analysis, Cambridge Tracts in Math. Cambridge University Press, Cambridge (2012)
12. Dineen, S.: The second dual of a JB*-triple system. In: Mujica, J. (ed.) Complex analysis. Functional Analysis and Approximation Theory, Amsterdam (1986)
13. Edwards, C.M., Ruttimann, G.T.: On the facial structure of the unit balls in a JBW*-triple and its predual. J. Lond. Math. Soc. **38**, 317–322 (1988)
14. Edwards, C.M., Rüttimann, G.T.: Compact tripotents in bi-dual JB*-triples. Math. Proc. Camb. Phil. Soc. **120**, 155–173 (1996)
15. Férnandez López, A., García Rus, E., Sánchez Campos, E., Siles Molina, M.: Strong regularity and generalized inverses in Jordan systems. Comm. Algbra **20**, 1917–1936 (1992)
16. Fernández-Polo, F.J., Peralta, A.M.: Closed tripotents and weak compactness in the dual space of a JB*-triple. J. Lond. Math. Soc. **74**(1), 75–92 (2006)
17. Fernández-Polo, F.J., Peralta, A.M.: Compact tripotents and the Stone-Weierstrass theorem for C*-algebras and JB*-triples. J. Oper. Theory **58**(1), 157–173 (2007)
18. Fernández-Polo, F.J., Peralta, A.M.: Weak compactness in the dual space of a JB^*-triple is commutatively determined. Math. Scand. **105**(2), 307–319 (2009)
19. Fernández-Polo, F.J., Peralta, A.M.: Non-commutative generalisations of Urysohn's lemma and hereditary inner ideals. J. Funct. Anal. **259**(2), 343–358 (2010)
20. Friedman, Y., Russo, B.: Structure of the predual of a JBW*-triple. J. Reine Angew. Math. **356**, 67–89 (1985)
21. Friedman, Y., Russo, B.: A Gelfand-Naimark theorem for JB*-triples. Duke Math. J. **53**, 139–148 (1986)
22. Hanche-Olsen, H., Størmer, E.: Jordan operator algebras, monographs and studies in mathematics 21. Pitman, London (1984)
23. Harris, L. A.: Bounded symmetric homogeneous domains in infinite dimensional spaces. In: Proceedings on Infinite Dimensional Holomorphy (Internat. Conf., Univ. Kentucky, Lexington, Ky., 1973), Lecture Notes in Math. Vol 364, pp. 13–40. Springer, Berlin (1974)
24. Harris, L. A.: A generalization of C*-algebras, Proc. London Math. Soc. (3). **42**(2), 331–361 (1981)
25. Jamjoom, F., Peralta, A.M., Siddiqui, A.A., Tahlawi, H.M.: Approximation and convex decomposition by extremals and the λ-function in JBW*-triples. Quart. J. Math. **66**, 583–603 (2015)
26. Jamjoom, F.B., Peralta, A.M., Siddiqui, A.A., Tahlawi, H.M.: Čebyšëv subspaces of JBW*-triples. J. Inequal. Appl. **2015**, 288 (2015). doi:10.1186/s13660-015-0813-2

27. Jamjoom, F.B., Siddiqui, A.A., Tahlawi, H.M.: On the geometry of the unit ball of a JB*-triple, Abstract and Applied Analysis JB*-triple. Abstr. Appl. Anal. **2013**, 8 (2013). doi:10.1155/2013/891249. (Article ID 891249)
28. Kadison, R.V., Ringrose, J.R.: Fundamental of the theory of operator algebras, Vol. I, American Mathematical Society, Providence, RI (1997)
29. Kaup, W.: A Riemann mapping theorem for bounded symmetric domains in complex banach spaces. Math. Z. **183**, 503–529 (1983)
30. Kaup, W.: On spectral and singular values in JB*-triples. Proc. Roy. Irish Acad. Sect. A **96**(1), 95–103 (1996)
31. Kaup, W.: On Grassmanian associated with JB*-triples. Math. Z. **236**, 567–584 (2001)
32. Kaup, W., Upmeier, H.: Jordan algebras and symmetric Siegel domains in banach spaces. Math. Z. **157**, 179–200 (1977)
33. Loos, O.: Bounded symmetric domains and Jordan pairs. University of California, Irvine, Math. Lectures (1977)
34. Pedersen, G.K.: Čebyšëv subspaces of a C*-algebra. Math. Scand. **45**, 147–156 (1979)
35. Robertson, A.G.: Best approximation in von Neumann algebras. Math. Proc. Camb. Philos. Soc. **81**, 233–236 (1977)
36. Robertson, A.G., Yost, D.: Chebyshev subspaces of operator algebras. J. Lond. Math. Soc. **2**(19), 523–531 (1979)
37. Singer, I.: Best approximation in normed linear spaces by elements of linear subspaces. Springer-Verlag, New York-Berlin (1970)
38. Takesaki, M.: Theory of operator algebras I. Springer, Berlin (1979)